12) Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy$

<u>Solution:</u> $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy$

$$= \int_1^e \int_1^{\log y} \left[\int_1^{e^x} \underset{u}{\log z} \, \underset{v}{\overset{\text{one}}{1}} \, dz \right] dx \, dy$$

$$= \int_1^e \int_1^{\log y} \left[\log z \; z - \int z \cdot \frac{1}{z} \, dz \right]_1^{e^x} dx \, dy$$

$\int dz = z$

$$= \int_1^e \int_1^{\log y} \left[z \log z - z \right]_1^{e^x} dx \, dy$$

$$= \int_1^e \int_1^{\log y} \left[\left(e^x \log e^x - e^x \right) - \left((1) \overset{0}{\cancel{\log 1}} - 1 \right) \right] dx \, dy$$

$$= \int_1^e \int_1^{\log y} \left[e^x (x-1) + 1 \right] dx \, dy$$

$$= \int_1^e \left[\int_1^{\log y} \left[e^x (x-1) + 1 \right] dx \right] dy$$

$$= \int_1^e \left[\left((x-1)e^x - e^x(1)\right) + x\right]_1^{\log y} dy$$

$$= \int_1^e \left[\left((x-1)e^x - e^x\right) + x\right]_1^{\log y} dy$$

$$= \int_1^e \left[(\log y - 1)e^{\log y} - e^{\log y} + \log y - \left((\cancel{1-1})^{0} e^1 - e^1 + 1\right)\right] dy$$

$$= \int_1^e \left[(\log y - 1)\, y - y + \log y - (-e+1)\right] dy$$

$$= \int_1^e \left[(\log y - 1)\, y - y + \log y + e - 1\right] dy$$

$$= \int_1^e \left[y\log y - y - y + \log y + e - 1\right] dy$$

$$= \int_1^e \left[y\,\underline{\log y} - 2y + \underline{\log y} + e - 1\right] dy$$

$$= \int_1^e \left[\log y\,(y+1) - 2y + e - 1\right] dy$$

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$$= \left[\log y \left(\frac{y^2}{2} + y\right) - \int_1^e \left(\frac{y^2}{2} + y\right) \frac{1}{y}\, dy - \left[\frac{2y^2}{2}\right] + ey - y \right]_1^e$$

$$= \left[\log y \left(\frac{y^2}{2} + y\right) - \int_1^e \left(\frac{y}{2} + 1\right) dy - y^2 + y(e-1) \right]_1^e$$

$$= \left[\log y \left(\frac{y^2}{2} + y\right) - \frac{y^2}{4} - y - y^2 + y(e-1) \right]_1^e$$

$$= \left[\left(\cancel{\log_e e}^{\,1} \left(\frac{e^2}{2} + e\right) - \frac{e^2}{4} - e - e^2 + e(e-1) \right) - \left(\cancel{\log 1}^{\,0} \left(\frac{1^2}{2} + 1\right) - \frac{1^2}{4} - 1 - 1^2 + 1(e-1) \right) \right]$$

$$= \left[\left(\frac{e^2}{2} + e\right) - \frac{5}{4} e^2 - e + e^2 - e - \left(-\frac{1}{4} - 1 - 1 + (e-1) \right) \right]$$

$$= \left[\frac{e^2}{2} + \cancel{e} - \frac{e^2}{4} - 2e + \frac{1}{4} + 2 \cancel{+ e} + 1 \right]$$

$$= \left[\frac{e^2}{2} - 2e + \frac{13}{4} \right] = \frac{2e^2 - 8e + 13}{4}$$

13) Evaluate $\iiint_R xyz\,dx\,dy\,dz$ over the region R enclosed by the co-ordinate plane and the plane $x+y+z=1$

Solution: In the given region, z varies from 0 to $1-x-y$ $[z=0,\quad z=\cancel{1}-x-y]$

For $z=0$, y varies from 0 to $1-x$.

For $y=0$, x varies from 0 to 1

$$\therefore \iiint_R xyz\,dx\,dy\,dz = \int_{x=0}^{1}\int_{y=0}^{1-x}\left[\int_{z=0}^{1-x-y} xyz\,dz\right]dy\,dx$$

$$= \int_0^1\int_0^{1-x}\left[xy\frac{z^2}{2}\right]_0^{1-x-y} dy\,dx$$

$$= \int_0^1\int_0^{1-x}\left[\frac{xy}{2}(1-x-y)^2 - 0\right]dy\,dx$$

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$$\int_0^1 \int_0^{1-x} \left[\frac{xy}{2}\left(1^2+x^2+y^2-2x+2xy-2y\right)\right] dy\, dx$$

$$\int_0^1 \frac{x}{2} \int_0^{1-x} y\left(1+x^2+y^2-2x+2xy-2y\right) dy\, dx$$

$$= \frac{1}{2}\int_0^1 x \int_1^{1-x} \left(y+yx^2+y^3-2xy+2xy^2-2y^2\right) dy\, dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{y^2}{2}+\frac{y^2x^2}{2}+\frac{y^4}{4}-\frac{\cancel{2}xy^2}{\cancel{2}}+\frac{2xy^3}{3}-\frac{2y^3}{3}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{y^2}{2}+\frac{y^2x^2}{2}+\frac{y^4}{4}-xy^2+\frac{2xy^3}{3}-\frac{2y^3}{3}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{y^4}{4}+y^2\left(\tfrac{1}{2}+\frac{x^2}{2}-x\right)+\frac{2}{3}y^3(x-1)\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{y^4}{4}+\frac{y^2}{2}\left(1+x^2-2x\right)+\frac{2}{3}y^3(x-1)\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{y^4}{4}+\frac{y^2}{2}(1-x)^2 - \frac{2}{3}y^3(\cancel{1-x})\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{(1-x)^4}{4} + \frac{(1-x)^2}{2}(1-x)^2 - \frac{2}{3}(1-x)^3(1-x) - 0\right]dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{(1-x)^4}{4} + \frac{(1-x)^4}{2} - \frac{2}{3}(1-x)^4\right]dx$$

$$= \frac{1}{2}\int_0^1 x(1-x)^4\left(1/4 + 1/2 - 2/3\right)dx$$

$$= \frac{1}{2}\int_0^1 \frac{x(1-x)^4}{12}\,dx.$$

$$= \frac{1}{24}\int_0^1 x(1-x)^4\,dx.$$

$$= \frac{1}{24}\left[x\left(-\frac{(1-x)^5}{5}\right) - \int_0^1 -\frac{(1-x)^5}{5}(1)\,dx\right]_0^1$$

$$= \frac{1}{24}\left[-\frac{x(1-x)^5}{5} + \left(-\frac{(1-x)^6}{6}\right)\frac{1}{5}\right]_0^1$$

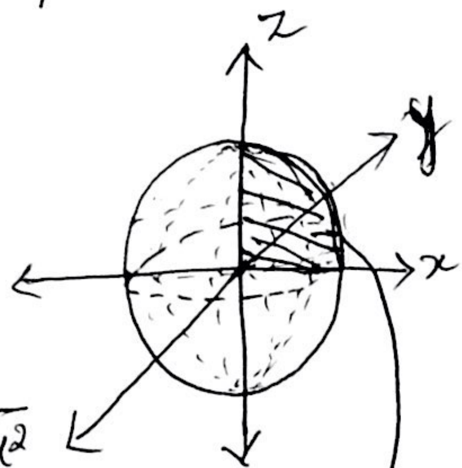
$$= \frac{1}{24}\left[-\frac{x(1-x)^5}{5} - \frac{(1-x)^6}{30}\right]_0^1$$

$$= \frac{1}{24}\left[\left(-1\,\frac{\cancel{(1-1)^5}}{5} - \frac{\cancel{(1-1)^6}}{30}\right) - \left(-0\,\frac{\cancel{(1-0)^5}}{5} - \frac{(1-0)^6}{30}\right)\right]$$

$$= \frac{1}{24}\left[\frac{1}{30}\right]$$

$$= \frac{1}{720}$$

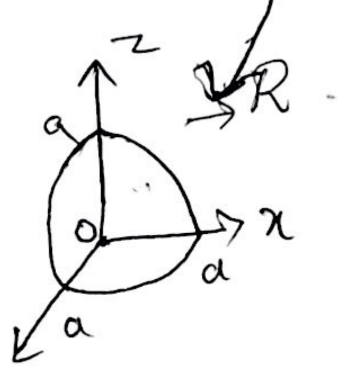
14) Evaluate $I = \iiint_R xyz \, dx\, dy\, dz$ where R is the positive octant of the sphere $x^2+y^2+z^2=a^2$



Solution: In the region, z varies from 0 to $\sqrt{a^2-x^2-y^2}$

For $z=0$, y varies from 0 to $\sqrt{a^2-x^2}$

For $z=0$, $y=0$, x varies from 0 to a



$$\iiint_R xyz \, dx\, dy\, dz = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xyz \, dz\, dy\, dx.$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\int_0^{\sqrt{a^2-x^2-y^2}} xyz \, dz \right] dy \, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[xy \frac{z^2}{2} \right]_0^{\sqrt{a^2-x^2-y^2}} dy \, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{xy}{2} \left[\left(\sqrt{a^2-x^2-y^2}\right)^2 - 0 \right] dy \, dx$$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} \frac{xy}{2} (a^2-x^2-y^2) \, dy \right] dx$$

$$= \int_0^a \frac{x}{2} \left[\int_0^{\sqrt{a^2-x^2}} (a^2y - x^2y - y^3) \, dy \right] dx$$

$$= \int_0^a \frac{x}{2} \left[\frac{a^2y^2}{2} - \frac{x^2y^2}{2} - \frac{y^4}{4} \right]_0^{\sqrt{a^2-x^2}} dx$$

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$$= \int_0^a \frac{x}{2}\left[\frac{a^2(\sqrt{a^2-x^2})^2}{2} - \frac{x^2(\sqrt{a^2-x^2})^2}{2} - \frac{(\sqrt{a^2-x^2})^4}{4}\right]dx$$

$$= \int_0^a \frac{x}{2}\left[\frac{a^2(a^2-x^2)}{2} - \frac{x^2(a^2-x^2)}{2} - \frac{(a^2-x^2)^2}{4}\right]dx$$

$$= \int_0^a \frac{x}{2}\left[\frac{(a^2-x^2)}{2} \cdot (a^2-x^2) - \frac{(a^2-x^2)^2}{4}\right]dx$$

$$= \int_0^a \frac{x}{2}\left[\frac{(a^2-x^2)^2}{2} - \frac{(a^2-x^2)^2}{4}\right]dx$$

$$= \int_0^a \frac{x}{2}\left[\frac{(a^2-x^2)^2}{4}\right]dx$$

$$= \frac{1}{8}\int_0^a x(a^2-x^2)^2\,dx$$

$$= \frac{1}{8}\int_0^a x\,(a^4+x^4-2a^2x^2)\,dx$$

$$= \frac{1}{8} \int_0^a \left(x a^4 + x^5 - 2a^2 x^3\right) dx$$

$$= \frac{1}{8} \left[\frac{x^2 a^4}{2} + \frac{x^6}{6} - \frac{\cancel{2}^{1} a^2 x^4}{\cancel{4}\, 2} \right]_0^a$$

$$= \frac{1}{8} \left[\frac{x^2 a^4}{2} + \frac{x^6}{6} - \frac{a^2 x^4}{2} \right]_0^a$$

$$= \frac{1}{8} \left[\frac{a^2 a^4}{2} + \frac{a^6}{6} - \frac{a^2 a^4}{2} - 0 \right]$$

$$= \frac{1}{8} \left[\cancel{\frac{a^6}{2}} + \frac{a^6}{6} - \cancel{\frac{a^6}{2}} \right]$$

$$= \frac{1}{8} \left[\frac{a^6}{6} \right]$$

$$= \frac{a^6}{48}$$